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Approximation of Integrable Functions by Wavelet Expansions

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Abstract. Walter (J Approx Theory 80:108–118, 1995), Xiehua (Approx Theory Appl 14(1):81–90, 1998) and Lal and Kumar (Lobachevskii J Math 34(2):163–172, 2013) established results on pointwise and uniform convergence of wavelet expansions. Working in this direction new more general theorems on degree of pointwise approximation by such expansions have been proved.

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1. Introduction

In this paper we use the following notations.

$L^p(\mathbb{R})$ $(1 < p < \infty)$ denotes the space of measurable, integrable with $p-th$ power functions f. The norm of $f \in L^p(\mathbb{R})$ is written by $\|f\|_{L^p(\mathbb{R})}$.

$l^2(\mathbb{Z})$ is the vector space of square-summable functions.

A multiresolution approximation of $L^2(\mathbb{R})$ (see [3]) is a sequence $(V_j)_{j\in Z}$ of closed subspaces of $L^2(\mathbb{R})$ such that the following hold:

$\forall_{j\in\mathbb{Z}} V_j \subset V_{j+1}$,
$\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \varnothing$,
$\forall_{j\in\mathbb{Z}} f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$,
$\forall_{j,k\in\mathbb{Z}} f(x) \in V_j \Leftrightarrow f(x-2^{-j}k) \in V_j$,

There exists an isomorphism I from V_0 onto $l^2(\mathbb{Z})$ which commutes with the action of $\mathbb{Z}$.

We consider orthonormal bases of wavelets in $L^2(\mathbb{R})$ (see [1]). These are functions of the form

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi\left(2^j x - k\right),$$

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for some fixed function ψ and are in turn based on a scaling function φ. In addition to the ladder of approximation spaces

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$$

we have the orthogonal complement of each in the next higher one on the ladders i.e.,

$$\cdots V_0 \oplus W_0 = V_1,\ V_1 \oplus W_1 = V_2 \cdots$$

The translates of the scaling function φ are an orthonormal basis of V_0 while the translates of ψ are an orthogonal basis of W_o (see [1]). The same holds for their dilations in V_j and W_j. Hence each $f \in L^2(\mathbb{R})$ has a representation

$$f(x) = f_n(x) + f_n^{\perp}(x),$$

where

$$f_n(x) = \sum_{k=-\infty}^{\infty} a_{n,k}\varphi_{n,k}(x), \quad \varphi_{n,k}(x) = 2^{\frac{n}{2}}\varphi(2^n x - k)$$

and

$$f_n^{\perp}(x) = \sum_{j=-n}^{\infty}\sum_{k=-\infty}^{\infty} b_{j,k}\psi_{j,k}(x), \quad \psi_{j,k}(x) = 2^{\frac{j}{2}}\psi\left(2^j x - k\right)$$

and

$$f(x) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} b_{j,k}\psi_{j,k}(x), \quad \text{with} \quad f_n(x) = \sum_{j=-\infty}^{n}\sum_{k=-\infty}^{\infty} b_{j,k}\psi_{j,k}(x),$$

where $a_{n,k}$, $b_{j,k}$ are expansion coefficients of f. f_n is said to be the partial sums of the wavelet expansion. It is given

$$f_n(x) = \int_{-\infty}^{\infty} q_n(x,t) f(t)\,dt = \int_{-\infty}^{\infty} p_n(x,t) f(t)\,dt,$$

by the reproducing kernels

$$q_n(x,t) = \sum_{k=-\infty}^{\infty} \varphi_{n,k}(x)\varphi_{n,k}(t) \text{ and } p_n(x,t) = \sum_{j=-\infty}^{n}\sum_{k=-\infty}^{\infty} \psi_{j,k}(x)\psi_{j,k}(t)$$

(see [4]).

The pointwise modules of continuity of the function f at point x are given by

$$\omega_x(f,\delta) = \sup_{|t|\le\delta} |f(t+x) - f(x)|,$$

$$w_x(f,\delta) = \frac{1}{\delta}\int_{-\delta}^{\delta} |f(t+x) - f(x)|\,dt,$$

for fixed x.

The deviation $f_n(x) - f(x)$ was estimated by Xiehua [11, Theorem 3.1, p. 84] (see also [5, Theorem 3.1] and [10]) as follows:

Theorem A. *Assume the scaling function φ satisfies the condition*

$$|\varphi(x)| \le \frac{C}{1+|x|^{\alpha}}, \quad \alpha > 1. \tag{1}$$

If $f \in L^2(\mathbb{R})$ is continuous at x, then we have

$$\begin{aligned} &|f_n(x) - f(x)| \\ &= O(1)\left\{\left(\|f\|_{L^2(\mathbb{R})} + \frac{1}{\alpha-1}|f(x)|\right)2^{-(\alpha-1)n} + \sum_{k=1}^{2^n} k^{-\alpha}\omega_x(f, k2^{-n})\right\}, \end{aligned}$$

where "O" depends only on C.

We also note the following well known result of Daubechies.

Theorem B [2, Theorem 9.1.6.]. *Let φ' be a continuous function such that φ and φ' satisfy (1). If $\varphi_{n,k}$ constitute an orthonormal basis on $L^2(\mathbb{R})$, then the $\{\varphi_{n,k} : n, k \in \mathbb{Z}\}$ also constitute an unconditional basis for all the spaces $L^p(\mathbb{R})$ $(1 < p < \infty)$.*

Here, we will give more general and precise estimates of the pointwise convergence of wavelet expansions.

2. Statement of the Results

We determine the degree of approximation of functions by wavelet expansions.

Theorem 1. *Assume that a continuous function φ' is such that φ and φ' satisfy (1). If $f \in L^p(\mathbb{R})$ $(1 < p < \infty)$, then*

$$|f_n(x) - f(x)| \le K\left\{2w_x(f, 2^{-n}) + \alpha 2^{n(1-\alpha)}\int_{2^{-n}}^{\infty} t^{-\alpha} w_x(f,t)dt\right\}$$

at every point x and with a positive constant K dependent on C.

Proof. From (1) it is easy to say that

$$|q(x,t)| \le \frac{K}{1+|x-t|^{\alpha}}. \tag{2}$$

Following Walter [10], Meyer [6] and Novikov, Protasov, Skopina [7]

$$\int_{-\infty}^{\infty} q_n(x,t)\,dt = 1, \quad \text{for } x \in \mathbb{R} \tag{3}$$

and we have

$$f_n(x) - f(x) = \int_{-\infty}^{\infty} q_n(x,t)\{f(t) - f(x)\}\,dt = \int_{-\infty}^{x-2^{-n}} + \int_{x-2^{-n}}^{x+2^{-n}} + \int_{x+2^{-n}}^{\infty}.$$

Next, using (2)

$$
\begin{aligned}
\left|\int_{x-2^{-n}}^{x+2^{-n}}\right| &\le \int_{x-2^{-n}}^{x+2^{-n}} \frac{K2^n}{1+(2^n|x-t|)^\alpha}\left|f(t)-f(x)\right|dt \\
&= \int_0^{2^{-n}} \frac{K2^n}{1+(2^n|t|)^\alpha}\left[\left|f(t+x)-f(x)\right|+\left|f(-t+x)-f(x)\right|\right]dt \\
&= \int_0^{2^{-n}} \frac{K2^n}{1+(2^n|t|)^\alpha}\left[\frac{d}{dt}\int_{-t}^{t}\left|f(u+x)-f(x)\right|du\right]dt \\
&= \left[\frac{K2^n}{1+(2^n|t|)^\alpha}\int_{-t}^{t}\left|f(u+x)-f(x)\right|du\right]_0^{2^{-n}} \\
&\quad + \int_0^{2^{-n}} \frac{\alpha K2^{n(1+\alpha)}|t|^{\alpha-1}}{\left[1+(2^n|t|)^\alpha\right]^2}\left[\int_{-t}^{t}\left|f(u+x)-f(x)\right|du\right]dt \\
&= \frac{K}{1+1^\alpha}w_x(f,2^{-n}) + w_x(f,2^{-n})\int_0^{2^{-n}}\frac{\alpha K2^{n\alpha}t^{\alpha-1}}{\left[1+(2^n t)^\alpha\right]^2}dt \\
&= Kw_x(f,2^{-n}).
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_{-\infty}^{x-2^{-n}} + \int_{x+2^{-n}}^{\infty}\right| &\le \left(\int_{-\infty}^{x-2^{-n}} + \int_{x+2^{-n}}^{\infty}\right)\frac{K2^n}{1+(2^n|x-t|)^\alpha}\left|f(t)-f(x)\right|dt \\
&\le \left(\int_{-\infty}^{-2^{-n}} + \int_{2^{-n}}^{\infty}\right)\frac{K2^n}{1+(2^n|t|)^\alpha}\left|f(t+x)-f(x)\right|dt \\
&\le K2^{n(1-\alpha)}\left(\int_{-\infty}^{-2^{-n}} + \int_{2^{-n}}^{\infty}\right)\frac{\left|f(t+x)-f(x)\right|}{|t|^\alpha}dt \\
&= K2^{n(1-\alpha)}\int_{2^{-n}}^{\infty} t^{-\alpha}\left[\frac{d}{dt}\int_{-t}^{t}\left|f(u+x)-f(x)\right|du\right]dt \\
&= K2^{n(1-\alpha)}\left[t^{-\alpha}\int_{-t}^{t}\left|f(u+x)-f(x)\right|du\right]_{2^{-n}}^{\infty} \\
&\quad +\alpha K2^{n(1-\alpha)}\int_{2^{-n}}^{\infty} t^{-\alpha-1}\left[\int_{-t}^{t}\left|f(u+x)-f(x)\right|du\right]dt.
\end{aligned}
$$

Observing that

$$
\begin{aligned}
&\lim_{t\to\infty} t^{-\alpha}\int_{-t}^{t}\left|f(u+x)-f(x)\right|du \\
&\quad \le \lim_{t\to\infty}\left\{2^{1-\frac{1}{p}}t^{-\alpha+1}\left[\frac{1}{t}\int_{-t}^{t}\left|f(u+x)\right|^p du\right]^{1/p} + 2t^{-\alpha+1}\left|f(x)\right|\right\} \\
&\quad \le \lim_{t\to\infty}\left\{2^{1-\frac{1}{p}}t^{-\alpha+1-\frac{1}{p}}\left\|f\right\|_{L^p(\mathbb{R})} + 2t^{-\alpha+1}\left|f(x)\right|\right\} = 0
\end{aligned}
$$

we obtain

$$\left|\int_{-\infty}^{x-2^{-n}}+\int_{x+2^{-n}}^{\infty}\right| \leq Kw_x(f,2^{-n})+K2^{n(1-\alpha)}\alpha\int_{2^{-n}}^{\infty}t^{-\alpha}w_x(f,t)dt.$$

Collecting our partial estimates the result follows. □

Corollary 1. *If $f\in\{g\in L^p(\mathbb{R}):w_x(g,\delta)=O\left(w_x(\delta)\right)\}$ where w_x is a function of modulus of continuity type, then by the monotonicity of $\frac{w_x(\delta)}{\delta}$ and $\alpha>2$, we obtain*

$$\left|f_n\left(x\right)-f\left(x\right)\right|=O\left(w_x(2^{-n})\right).$$

Theorem 2. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned}\left|f_n\left(x\right)-f\left(x\right)\right| &\leq 2Kw_x(f,2^{-n})+K\alpha2^{n(1-\alpha)}\int_{2^{-n}}^{1}t^{-\alpha}w_x(f,t)dt\\ &+K2^{n(1-\alpha)}\left[\left(\frac{2}{(\alpha q-1)^{1/q}}+2^{\frac{1}{q}}\right)\|f\|_{L^p(\mathbb{R})}\right.\\ &\left.+2\left(\frac{1}{\alpha-1}+1\right)\left|f\left(x\right)\right|\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Similarly as above

$$f_n\left(x\right)-f\left(x\right)=\int_{-\infty}^{\infty}q_n\left(x,t\right)\left\{f\left(t\right)-f\left(x\right)\right\}dt=\int_{-\infty}^{x-2^{-n}}+\int_{x-2^{-n}}^{x+2^{-n}}+\int_{x+2^{-n}}^{\infty}$$

and

$$\left|\int_{x-2^{-n}}^{x+2^{-n}}\right|\leq Kw_x(f,2^{-n}).$$

Further

$$\begin{aligned}&\left|\int_{-\infty}^{x-2^{-n}}+\int_{x+2^{-n}}^{\infty}\right|\\ &\leq K2^{n(1-\alpha)}\int_{2^{-n}}^{\infty}\frac{\left|f\left(t+x\right)-f\left(x\right)\right|+\left|f\left(-t+x\right)-f\left(x\right)\right|}{t^{\alpha}}dt\leq\int_{2^{-n}}^{1}+\int_{1}^{\infty}\end{aligned}$$

and

$$\begin{aligned}\int_{2^{-n}}^{1} &= K2^{n(1-\alpha)}\int_{2^{-n}}^{1}t^{-\alpha}\left[\frac{d}{dt}\int_{-t}^{t}\left|f\left(u+x\right)-f\left(x\right)\right|du\right]dt\\ &= K2^{n(1-\alpha)}\left[t^{-\alpha}\int_{-t}^{t}\left|f\left(u+x\right)-f\left(x\right)\right|du\right]_{2^{-n}}^{1}\\ &\quad+\alpha K2^{n(1-\alpha)}\int_{2^{-n}}^{1}t^{-\alpha-1}\left[\int_{-t}^{t}\left|f\left(u+x\right)-f\left(x\right)\right|du\right]dt\end{aligned}$$

$$\leq K2^{n(1-\alpha)}\left[2^{1-\frac{1}{p}}\left\|f\right\|_{L^{p}(\mathbb{R})}+2\left|f\left(x\right)\right|\right]+Kw_{x}(f,2^{-n})$$
$$+K2^{n(1-\alpha)}\alpha\int_{2^{-n}}^{1}t^{-\alpha}w_{x}(f,t)dt,$$
$$\int_{1}^{\infty}\leq K2^{n(1-\alpha)}\left\{\left(\int_{1}^{\infty}t^{-q\alpha}dt\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}}\left|f\left(t+x\right)\right|^{p}dt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_{1}^{\infty}t^{-q\alpha}dt\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}}\left|f\left(-t+x\right)\right|^{p}dt\right)^{\frac{1}{p}}+2\left|f\left(x\right)\right|\int_{1}^{\infty}t^{-\alpha}dt\right\}$$
$$\leq K2^{n(1-\alpha)}\left(\frac{2}{\left(\alpha q-1\right)^{1/q}}\left\|f\right\|_{L^{p}(\mathbb{R})}+\frac{2\left|f\left(x\right)\right|}{\alpha-1}\right).$$

Thus we obtain the desired estimate. □

Corollary 2. *Under the assumptions of Theorem 1 and if* $w_x(f,\delta)=O\left(\delta^{\gamma}\right)$, *then by Theorem 2*

$$\left|f_n\left(x\right)-f\left(x\right)\right|=\begin{cases}O\left(2^{-n\gamma}\right) & \text{when } 0<\gamma<\alpha-1,\\ O\left(2^{-n(\alpha-1)}\right) & \text{when } \gamma>\alpha-1,\\ O\left(n2^{-n(\alpha-1)}\right) & \text{when } \gamma=\alpha-1.\end{cases}$$

The same order of approximation one can find in the paper of Skopina [9].

Remark 1. If x is a point of continuity of $f\in L^p(\mathbb{R})$, then $\omega_x(f,\delta)=o\left(1\right)$, whence $w_x(f,\delta)=o\left(1\right)$ and thus

$$\left|f_n\left(x\right)-f\left(x\right)\right|=o\left(1\right).$$

We also show our estimates in the following summation forms.

Theorem 3. *Under the assumptions of Theorem 1*

$$\left|f_n\left(x\right)-f\left(x\right)\right|\leq K_{\alpha}2^{n(1-\alpha)}\sum_{\nu=0}^{2^n-2}\left(\nu+2\right)^{\alpha-2}w_x\left(f,\frac{1}{\nu+1}\right)$$
$$+2\alpha K\sum_{\nu=2^n}^{\infty}\nu^{-\alpha}w_x\left(f,\frac{\nu+1}{2^n}\right),$$

where $K_{\alpha}=K\left[2^{\alpha+2}+2\alpha\right]$.

Proof. Since $\delta w_x(f,\delta)$ is nondecreasing function of δ, therefore for $N\geq 1$

$$\int_{2^{-n}}^{N}t^{-\alpha}w_x(f,t)dt=\left(\int_{2^{-n}}^{1}+\int_{1}^{N}\right)t^{-\alpha}w_x(f,t)dt$$
$$=\int_{1}^{2^n}u^{\alpha-1}\frac{1}{u}w_x\left(f,\frac{1}{u}\right)du+\sum_{\nu=2^n}^{N2^n-1}\int_{\nu2^{-n}}^{(\nu+1)2^{-n}}t^{-\alpha-1}tw_x(f,t)dt$$

$$
\begin{aligned}
&= \sum_{\nu=1}^{2^n-1} \int_{\nu}^{\nu+1} u^{\alpha-1} \frac{1}{u} w_x \left(f, \frac{1}{u}\right) du + \sum_{\nu=2^n}^{N2^n-1} \int_{\nu 2^{-n}}^{(\nu+1)2^{-n}} t^{-\alpha-1} t w_x(f,t) dt \\
&\leq \sum_{\nu=1}^{2^n-1} \frac{1}{\nu} w_x \left(f, \frac{1}{\nu}\right) \left[\frac{u^{\alpha}}{\alpha}\right]_{u=\nu}^{\nu+1} + \sum_{\nu=2^n}^{N2^n-1} \frac{\nu+1}{2^n} w_x \left(f, \frac{\nu+1}{2^n}\right) \left[\frac{t^{-\alpha}}{-\alpha}\right]_{t=\nu 2^{-n}}^{(\nu+1)2^{-n}} \\
&\leq \sum_{\nu=1}^{2^n-1} \frac{1}{\nu} w_x \left(f, \frac{1}{\nu}\right) (\nu+1)^{\alpha-1} + \sum_{\nu=2^n}^{N2^n-1} \frac{\nu+1}{2^n} w_x \left(f, \frac{\nu+1}{2^n}\right) \frac{\nu^{-\alpha-1}}{2^{-\alpha n}} \\
&\leq 2 \sum_{\nu=1}^{2^n-1} w_x \left(f, \frac{1}{\nu}\right) (\nu+1)^{\alpha-2} + 2^{n(\alpha-1)+1} \sum_{\nu=2^n}^{N2^n-1} \nu^{-\alpha} w_x \left(f, \frac{\nu+1}{2^n}\right) \\
&\leq 2 \sum_{\nu=0}^{2^n-2} (\nu+2)^{\alpha-2} w_x (f, \frac{1}{\nu+1}) + 2^{n(\alpha-1)+1} \sum_{\nu=2^n}^{N2^n-1} \nu^{-\alpha} w_x \left(f, \frac{\nu+1}{2^n}\right) \\
&\leq 2 \sum_{\nu=0}^{2^n-2} (\nu+2)^{\alpha-2} w_x \left(f, \frac{1}{\nu+1}\right) + 2^{n(\alpha-1)+1} \sum_{\nu=2^n}^{\infty} \nu^{-\alpha} w_x \left(f, \frac{\nu+1}{2^n}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\nu=0}^{2^n-2} (\nu+2)^{\alpha-2} w_x \left(f, \frac{1}{\nu+1}\right) \\
&\quad \geq \sum_{\nu=2^{n-1}-1}^{2^n-2} (\nu+2)^{\alpha-1} \frac{1}{\nu+2} w_x \left(f, \frac{1}{\nu+1}\right) \\
&\quad \geq \frac{1}{2} \sum_{\nu=2^{n-1}-1}^{2^n-2} (\nu+2)^{\alpha-1} \frac{1}{\nu+1} w_x \left(f, \frac{1}{\nu+1}\right) \\
&\quad \geq \frac{1}{2} \sum_{\nu=2^{n-1}-1}^{2^n-2} (\nu+2)^{\alpha-1} \frac{1}{2^n} w_x \left(f, \frac{1}{2^n}\right) \geq \frac{1}{2} \frac{1}{2^n} w_x \left(f, \frac{1}{2^n}\right) \left(2^{n-1}+1\right)^{\alpha-1} 2^{n-1} \\
&\quad \geq \frac{1}{2} \frac{1}{2^n} w_x \left(f, \frac{1}{2^n}\right) 2^{(n-1)\alpha} = \frac{1}{2^{\alpha+1}} 2^{n(\alpha-1)} w_x \left(f, \frac{1}{2^n}\right).
\end{aligned}
$$

Thus, by Theorem 1 our result follows. $\square$

Theorem 4. *Under the assumptions of Theorem 1*

$$
\begin{aligned}
|f_n(x) - f(x)| \leq K_\alpha \sum_{\nu=1}^{2^n-1} \nu^{-\alpha} w_x \left(f, \frac{\nu+1}{2^n}\right) \\
+K 2^{n(1-\alpha)} \left[\left(\frac{2}{(\alpha q-1)^{1/q}} + 2^{\frac{1}{q}} \right) \|f\|_{L^p(\mathbb{R})} \right. \\
\left. +2 \left(\frac{1}{\alpha-1} + 1 \right) |f(x)| \right],
\end{aligned}
$$

where $K_\alpha = K\left[2^{\alpha+2} + 2\alpha\right].$

Proof. Using the monotonicity of $\delta w_x(f,\delta)$ we obtain that

$$\begin{aligned}
\int_{2^{-n}}^{1} t^{-\alpha} w_x(f,t)dt &= \sum_{\nu=1}^{2^n-1} \int_{\nu 2^{-n}}^{(\nu+1)2^{-n}} t^{-\alpha-1} t w_x(f,t)dt \\
&\le \sum_{\nu=1}^{2^n-1} \frac{\nu+1}{2^n} w_x\left(f,\frac{\nu+1}{2^n}\right) \int_{\nu 2^{-n}}^{(\nu+1)2^{-n}} t^{-\alpha-1}dt \\
&= \sum_{\nu=1}^{2^n-1} \frac{\nu+1}{2^n} w_x\left(f,\frac{\nu+1}{2^n}\right) \left[\frac{t^{-\alpha}}{-\alpha}\right]_{t=\nu 2^{-n}}^{(\nu+1)2^{-n}} \\
&= \frac{1}{-\alpha} \sum_{\nu=1}^{2^n-1} \frac{\nu+1}{2^n} w_x(f,\frac{\nu+1}{2^n}) \left[\left(\frac{2^n}{\nu+1}\right)^{\alpha} - \left(\frac{2^n}{\nu}\right)^{\alpha}\right] \\
&= \frac{1}{\alpha} \sum_{\nu=1}^{2^n-1} \frac{\nu+1}{2^{(1-\alpha)n}} w_x\left(f,\frac{\nu+1}{2^n}\right) \left[\nu^{-\alpha} - (\nu+1)^{-\alpha}\right] \\
&\le 2^{(\alpha-1)n} \sum_{\nu=1}^{2^n-1} (\nu+1)\,\nu^{-\alpha-1} w_x\left(f,\frac{\nu+1}{2^n}\right) \\
&\le 2^{(\alpha-1)n+1} \sum_{\nu=1}^{2^n-1} \nu^{-\alpha} w_x\left(f,\frac{\nu+1}{2^n}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\nu=1}^{2^n-1} \nu^{-\alpha} w_x\left(f,\frac{\nu+1}{2^n}\right) \\
&\quad \ge 2^n \sum_{\nu=1}^{2^n-1} (\nu+1)^{-1-\alpha} \frac{\nu+1}{2^n} w_x\left(f,\frac{\nu+1}{2^n}\right) \ge w_x\left(f,\frac{1}{2^n}\right) \sum_{\nu=1}^{2^n-1} (\nu+1)^{-1-\alpha} \\
&\quad \ge (1+1)^{-1-\alpha} w_x\left(f,\frac{1}{2^n}\right) = 2^{-1-\alpha} w_x\left(f,\frac{1}{2^n}\right).
\end{aligned}$$

Hence, by Theorem 2, analogously as above, our result follows. □

Remark 2. Since

$$w_x(f,\delta) \le 2\omega_x(f,\delta), \tag{4}$$

we can immediately write all of the above estimations with ω_x instead of w_x.

Remark 3. In the case $p=2$ our assumptions confine to the function φ only and thus the mentioned result of Xiehua [11, Theorem 3.1, p. 84] follows immediately from Theorem 4, by Remark 2.

Remark 4. The convergence at Lebesgue's and strong Lebesgue's points were investigated in the papers of Skopina [8] and of Kelly, Kon, Raphael [4].

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